STOR642 - Comprehensive Written Exam - August 2023

This test consists of four questions. The relative weights are specified following each part. You are NOT allowed to refer to any notes, books etc.

Good luck!

1− Customers arrive at a service facility according to a Poisson process with rate λ. The facility consists of two stations in series. Customers first queue in the first station and after they receive service they either immediately join the queue in the second station with probability 0.5 or they rejoin the queue in the first station with the remaining probability. After receiving service in the second station, customers either leave right away with probability 0.5 or they join the queue of the **first** station with the remaining probability. There is a single server in each station. Service times in the first station are independent and identically distributed with an exponential distribution with mean m_1 and service times in the second station are independent and identically distributed with an exponential distribution with mean m_2. (Service times in both stations are also independent of the arrival process, service times in the other station, queue lengths in both stations and customers' routing probabilities.)

- **(a)** Consider the stochastic process that keeps track of the number of customers in each station. What kind of a process is this? **(3 points)**
- **(b)** State the stability condition, i.e., the condition under which the limiting distribution for the number of customers in each station exists. **(8 points)**
- **(c)** Assuming stability, give an expression for the limiting joint distribution for the number of customers in each station. **(9 points)**

2− Consider a service facility that is so popular that the provider limits the total number of customers in the system at any point in time by $N < \infty$ and as soon as a customer leaves the system a new customer is admitted right away. Similar to the setting in Question 1, the service facility consists of two stations in series and the newly admitted customers join the first station. However, unlike the setting of Question 1, any customer who finishes service in the first station joins the second station right away and any customer who finishes service in the second station leaves the system right away.

- **(a)** Suppose that there is a single server in each station and service times in station i (for $i = 1, 2$) are independent and identically distributed with an exponential distribution with mean m_i. Do you need a stability condition so that the limiting joint distribution for the number of customers in each station exists? If yes, specify the condition. If not, explain why no condition is needed. **(3 points)**
- **(b)** Give an expression for the limiting joint distribution for the number of customers in each station. **(8 points)**
- **(c)** Suppose now that for $i = 1, 2$, service times in station i are not necessarily exponentially distributed and there are N servers in each station. In this case, is it possible to determine the limiting distribution for the number of customers in each station? If yes, provide the distribution; if not, explain why not. **(9 points)**

3– Suppose that a machine is inspected regularly and if it is found to be in a failed state it is repaired immediately. (When a machine fails, it remains in the failed state until the next inspection.) The times between consecutive inspections are independent and identically distributed with an exponential distribution with mean $1/\mu$. The lifetimes of machines are independent and identically distributed with an exponential distribution with mean $1/\lambda$.

- **(a)** Let $N(t)$ denote the number of failures by time t. Is $\{N(t), t \geq 0\}$ a renewal process? Find $\lim_{t\to\infty} \frac{N(t)}{t}$. **(5 points)**
- **(b)** Let $Z(t) = 0$ if the machine is in the failed state and $Z(t) = 1$ if the machine is operational at time t. Write a renewal-type equation for $P\{Z(t) = 1\}$. **(7 points)**
- **(c)** Using part (b) and the key renewal theorem, find $\lim_{t\to\infty} P\{Z(t) = 1\}$. **(8 points)**
- **(d)** Using part (b) and the solution to the renewal-type equation, give an expression for $P\{Z(t) = 1\}$ for $t \geq 0$. **(7 points)**
- **(e)** Suppose that the machine generates revenue of \$$r$ per unit time it is working. It costs \$$c_1$ to repair each failed machine and \$$c_2$ to make an inspection. What is the long-run net profit per unit time?**(8 points)**

4– Suppose that two players, A and B, are repeatedly solving puzzles. As soon as one of the players solves a puzzle, the other player stops working on that puzzle. Then, they take a break and as soon as the break is over, they start working on the next puzzle. Whoever solves a puzzle first gets one point. As soon as either one of the players accumulates two more points than the other one, they reset their scores back to zero. (This means that it is sufficient to keep track of the difference in the accumulated points rather than the accumulated points by each player separately.) The times it takes player A to solve the puzzles are independent and identically distributed with an exponential distribution with mean $1/\lambda_1$ and the times it takes player B to solve the puzzles are independent and identically distributed with an exponential distribution with mean $1/\lambda_2$. Assume that the times it takes any given player to solve the puzzles are also independent of the times for the other player. Break times depend on who won the last point but otherwise are independent of everything else. If the last point was won by player A, then the duration of the break is determined by the cumulative distribution function $B_1(\cdot)$. If the last point was won by player B, then the duration of the break is determined by the cumulative distribution function $B_2(\cdot)$. Both $B_1(\cdot)$ and $B_2(\cdot)$ are continuous functions.

- **(a)** Model this system as a semi-Markov process. Provide the state space clearly explaining what each state represents and the kernel. **(10 points)**
- **(b)** Is the semi-Markov process you described in part (a) irreducible? Is it aperiodic? Why or why not? **(7 points)**
- **(c)** Explain how you would determine the limiting probability that the players are on a break. DO NOT SOLVE for the limiting probabilities. Just explain how it can be determined using your answers to parts (b) and (c). **(8 points)**