August 2023 Name:________________________

Comprehensive written exam – STOR654 Mathematical Statistics

Unless otherwise noted, all problem parts have equal weight. In budgeting your time expect that some part will take longer than others. When solving multi-part problems feel free to use results of earlier parts even if you cannot solve them in proving later parts.

1. Let $F(x)$ be cumulative distribution function (CDF) and define $F^{-}(u) = \inf\{x : F(x) \geq u\}$.
   (a) Show $F(x) \geq u$ if and only if $F^{-}(u) \leq x$.
   (b) Prove or disprove: If $U \sim U(0,1)$, then $X = F^{-}(U)$ is a random variable with CDF F.

2. Let $X_1, \dots, X_n$ be i.i.d discrete random variables with probability mass function $f(x|p) = p(1-p)^x I_{\{0,1,2,\dots\}}(x)$. (Hint: $P(X_i \geq s) = (1-p)^s$ for $s \in \{0, 1, 2, \dots\}$.)
   (a) Find the mean, variance, and moment generating function of X_i.
   (b) What is the distribution of the minimum $X_{(1)}$?
   (c) Prove or disprove $P(\bar{X}_n \geq a) \leq \left((a+1)p\left(\frac{(a+1)(1-p)}{a}\right)^a\right)^n$, where $\bar{X}_n$ is the sample mean and $a > \frac{1-p}{p}$.
   (d) Propose a 95% confidence interval for p. Evaluate for $n = 1, X_1 = 10$.
   (e) Assuming that p has prior Beta(α, β) prior, find the posterior distribution of p?
   (f) What is the Bayes rule (estimator) using the prior from Part 2e and loss function $L(a,p) = \frac{p^2(a-p)^2}{1-p}$?

3. Consider $X_1, \dots, X_n$ i.i.d from Gamma(a, a), $a > 0$, i.e. $f(x|a) = \frac{x^{a-1}a^a e^{-ax}}{\Gamma(a)} I_{(0,\infty)}(x)$.
   (a) Find a method of moments estimator of a.

(b) Find the maximum likelihood estimator of a. (Hint: There might not be a closed form solution. Please state which equation would you solve numerically.)

(c) Find minimal sufficient statistics for a. Is it complete?

(d) Define $Y = \log(X) - X$. What is the support of Y? Calculate the density of Y. Is it a member of exponential family? (Hint: You may need to define your own functions as roots of an equation.)

(e) For this problem part consider $n = 1$. Find the UMP test for testing $\mathcal{H}_0 : a = 1$ versus $\mathcal{H}_1 : a < 1$. What will be your decision at the $\alpha = 0.005$ level if $X = 1$?