COMPREHENSIVE WRITTEN EXAMINATION, PAPER III
PART 1: FRIDAY AUGUST 18, 2023 9:00 A.M.–11:00 A.M.
STOR 664 Theory Question (50 points)

This is a closed-book exam: no access to course materials or other (e.g. internet) resources is allowed. Answers in a blue book (provided). No communication is allowed with individuals either inside or outside the exam room; however, if you have queries about the exam, you may call or text the instructor at the phone number provided.

Many environmental variables display both seasonal variation and trends. A geophysicist collects monthly data for m years on an environmental variable Y and looks for evidence of both an annual sinusoidal effect and a linear trend. A plausible model for this purpose is

$$Y_t \;=\; \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \beta_3 x_{3t} + \epsilon_t,\ 1 \le t \le 12m, \tag{1}$$

where Y_t is the observation in month t and $x_{1t} = t - \frac{12m+1}{2},\ x_{2t} = \cos\left(\frac{2\pi t}{12}\right),\ x_{3t} = \sin\left(\frac{2\pi t}{12}\right)$, and the errors ϵ_t are independent $\mathcal{N}[0, \sigma^2]$ as in the usual linear model assumptions. Note that with these definitions, each of $\sum_{t=1}^{12m} x_{jt} = 0,\ j = 1, 2, 3$.

In the following questions, you are asked to derive algebraic formulas for several estimators or tests from this model. Any algebraically correct answer will earn positive credit, but answers that show how to reduce the expressions to their simplest forms will earn the greatest credit.

(a) Write the above model in the form $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, and show explicitly how $\mathbf{y},\ X,\ \boldsymbol{\beta}$ and $\boldsymbol{\epsilon}$ are derived from the quantities in (1). In particular, give expressions for the matrices X, X^TX and $(X^TX)^{-1}$ in the simplest form you can derive, using the formula sheet at the end of this exam. (*Note:* For $(X^TX)^{-1}$ and in subsequent calculations that depend on $(X^TX)^{-1}$, it will suffice to give the answer in terms of the quantities a, b, c, d, e from the formula sheet, provided you explain clearly how these numbers are calculated.) **[12 points].**

(b) Using your results from (a), given explicit expressions for the least squares estimators $\hat{\beta}_j,\ j = 0, 1, 2, 3$, and give formulas for their standard error. **[8 points].**

(c) The geophysicist is convinced that there is periodic variation in the time series, but is not sure about the trend. Show how to construct a hypothesis test of $H_0 :\ \beta_1 = 0$ against the alternative $H_1 :\ \beta_1 \neq 0$, where $\beta_0,\ \beta_2$ and β_3 and the common sample variance σ^2 are unknown. Assume a two-sided test with significance level 0.05. **[6 points].**

(d) Alternatively, suppose the geophysicist accepts the existence of a trend but is unsure about the periodic component. This suggests testing $H_0 :\ \beta_2 = \beta_3 = 0$ against the alternative H_1 that at least one of β_2 or β_3 is non-zero, again assuming $\beta_0,\ \beta_1$ and σ^2 are unrestricted. Describe the steps needed to construct a hypothesis test of H_0 against H_1. Assume a significance level of 0.05. **[12 points].**

(e) What is the power of the test in (d)? Describe explicitly the steps needed to calculate the power when $\beta_2,\ \beta_3$ and σ^2 are given. **[12 points].**

Formula Sheet

You may assume any of the following without proof.

$$
\begin{aligned}
\sum_{k=1}^{12m}\left(k-\frac{12m+1}{2}\right) &= 0,\\
\sum_{k=1}^{12m}\left(k-\frac{12m+1}{2}\right)^2 &= m(12m+1)(12m-1),\\
\sum_{k=1}^{12m}\cos\left(\frac{2\pi k}{12}\right) &= 0,\\
\sum_{k=1}^{12m}\sin\left(\frac{2\pi k}{12}\right) &= 0,\\
\sum_{k=1}^{12m}\cos^2\left(\frac{2\pi k}{12}\right) &= 6m,\\
\sum_{k=1}^{12m}\sin^2\left(\frac{2\pi k}{12}\right) &= 6m,\\
\sum_{k=1}^{12m}\cos\left(\frac{2\pi k}{12}\right)\sin\left(\frac{2\pi k}{12}\right) &= 0,\\
\sum_{k=1}^{12m}\left(k-\frac{12m+1}{2}\right)\cos\left(\frac{2\pi k}{12}\right) &= 6m,\\
\sum_{k=1}^{12m}\left(k-\frac{12m+1}{2}\right)\sin\left(\frac{2\pi k}{12}\right) &= -12m\left(1+\frac{\sqrt{3}}{2}\right).
\end{aligned}
$$

The inverse of the matrix $\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & C & D \\ 0 & C & C & 0 \\ 0 & D & 0 & C \end{pmatrix}$ is of the form $\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & -b & c \\ 0 & -b & d & -c \\ 0 & c & -c & e \end{pmatrix}$ where

$$
\begin{aligned}
a &= \frac{1}{A},\\
b &= -\frac{C}{D^2+C^2-BC},\\
c &= \frac{D}{D^2+C^2-BC},\\
d &= \frac{D^2-CB}{C(D^2+C^2-BC)},\\
e &= \frac{C-B}{D^2+C^2-BC}.
\end{aligned}
$$