STOR 642
Comprehensive Written Examination
August 2011

This test consists of two questions.
This is a closed book exam.
Explain your answers in detail.
The duration of the exam is 2 hours.
The relative weights are given in the parentheses.

Problem 1. Patients arrive at a clinic one by one. The \( n \)th patient at a clinic is scheduled to arrive time \( nd \), \( n = 0, 1, 2, \ldots \), where \( d > 0 \) is a fixed number. The \( n \)th patient fails to show up for his/her appointment with probability \( \theta \), independently of all other patients. There is a single doctor who sees the patients in a first come first served fashion. The service times are iid \( \text{Exp}(\mu) \). Let \( X(t) \) be the number of customers in the clinic at time \( t \). Let \( X_n \) be the number of patients in the clinic just before the \( n \)th scheduled arrival time, and \( Y_n \) be the number of patients in the clinic as seen by the \( n \) patient who actually arrives.

1. (5) Show that the following limit exists:

\[
p_j = \lim_{n \to \infty} P(X_n = j), \quad j \geq 0.
\]

Compute it.

2. (5) Show that the following limit exists:

\[
q_j = \lim_{n \to \infty} P(Y_n = j), \quad j \geq 0.
\]

Compute it.

3. (2) Define a Markov regenerative process. Show that \( \{X(t), t \geq 0\} \) is a Markov regenerative process.

4. (5) Show that the following limit exists:

\[
r_j = \lim_{t \to \infty} P(X(t) = j), \quad j \geq 0.
\]

Compute it.

5. (3) What is the relationship among the \( p_j \)'s, \( q_j \)'s and \( r_j \)'s?

6. (5) Suppose the waiting costs of the patients is \( w \) per unit time, and the idle time of the doctor costs \( h \) per unit time. Compute the long run cost per unit time as a function of \( d \).
Problem 2. (You may use the fact given at the end of this problem if needed.)

1. (1) Define a standard Brownian.

2. (2) Define a renewal process. Define transience and recurrence of a renewal process. Give the criteria for transience and recurrence.

3. (4) Let \( \{B(t), t \geq 0\} \) be a standard Brownian motion. Define \( S_0 = 0 \) and

\[
S_{n+1} = \min\{t > S_n : B(t) \in \{B(S_n) - 1, B(S_n) + 1\}\}, \ n \geq 0.
\]

Let

\[
N(t) = \sup\{n \geq 0 : S_n \leq t\}. \quad (1)
\]

Show that \( \{N(t), t \geq 0\} \) is a renewal process. Is it transient or recurrent?

4. (4) State and prove the almost sure version of the elementary renewal theorem for a recurrent renewal process.

5. (4) Compute \( \lim_{t \to \infty} N(t)/t \), where \( N(t) \) is as defined in Equation 1.

6. (2) Compute \( E(\int_0^{S_1} B^2(u) du | B(0) = 0) \).

7. (1) Define a renewal reward process.

8. (4) Define \( R(t) = \int_0^t [B(u) - B(T_{N(u)})]^2 du \). Show that \( \{R(t), t \geq 0\} \) is a renewal reward process.

9. (3) Compute \( \lim_{t \to \infty} R(t)/t \).

**Fact:** Let \( \{B(t), t \geq 0\} \) be a standard Brownian motion, \( T_{ab} = \min\{t \geq 0 : B(t) \in \{a, b\}\} \). Let \( c(x) = E(\int_0^{T_{ab}} f(B(u)) du | B(0) = x) \) for \( a \leq x \leq b \). Then \( c \) satisfies the differential equation

\[
\frac{1}{2}c''(x) = -f(x),
\]

with boundary condition \( c(a) = c(b) = 0 \).