General Instructions

This examination is closed-book, and consists of three questions. Answer all three as clearly and concisely as you are able. Use of the internet and/or mobile devices is not permitted.
Question 1. (33 points)

a. (11 points) For all positive integers \( n \) describe a directed graph with \( n \) nodes, which has at least \( 2^{cn} \) distinct directed cycles, for some fixed \( c > 0 \).

(There is no need to optimize \( c \); any \( c > 0 \) will do.)

b. (22 points) You are given a directed graph \( G = (N, A) \) with \( n \) nodes and \( m \) arcs.

Describe an integer programming problem which determines the minimum number of arcs that must be removed from \( G \) so the remaining graph has no directed cycle.

The IP should have \( O(n + m) \) variables and constraints; the range of all variables (i.e. the difference between their upper and lower bounds) should also be \( O(n + m) \).

Carefully describe in words the meaning of all variables and constraints.

(Hint: a directed graph has no directed cycle, if and only if it has a . . . )
Question 2. (33 points) Suppose that $B \in \mathbb{R}^{m \times m}$ is a positive semi-definite matrix (i.e., $x^T B x \geq 0$ for all $x \in \mathbb{R}^m$). Let $b \in \mathbb{R}^m$ be a given vector. Consider the set

$$S = \{ y \in \mathbb{R}^m \mid By \geq b, \ y \geq 0 \}.$$ 

Suppose that $S$ is nonempty. Prove that $S$ is an unbounded set.

Hint: A subset $S$ of an Euclidean space is said to be bounded, if there exists a positive real number $M$ such that every $x \in S$ satisfies $\|x\|_2 \leq M$. We say $S$ is unbounded, if it is not bounded. You can use the fact that $S$ is unbounded if and only if there exists a vector $c \in \mathbb{R}^m$ such that the LP $\max_{y \in S} c^T y$ is an unbounded LP.

Question 3. (34 points) Consider the following linear program in canonical form, given in a simplex tableau.

\[
\begin{array}{cccccc|c|c}
\text{z} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{rhs} & \text{basic var} \\
1 & 4 & -3 & -2 & 0 & 0 & 0 & 10 & z = 10 \\
0 & 0.6 & -0.5 & -0.4 & 1 & 0 & 0 & 2 & x_4 = 2 \\
0 & -0.2 & 0.8 & -0.5 & 0 & 1 & 0 & 3 & x_5 = 3 \\
0 & -0.3 & -0.2 & 1 & 0 & 0 & 1 & 4 & x_6 = 4 \\
\end{array}
\]

Prove the following statements.

a. (12 points) In any feasible solution $x$ of the above linear program, at least one of the two components $x_1$ and $x_4$ is strictly positive.

b. (22 points) In any basic feasible solution $x$ of the above linear program, one of the two components $x_1$ and $x_4$ is exactly zero. (Hint: look at other pairs of variables as well.)