Instructions: All problems have equal weight; partial credit will be given for each part of a problem. In some cases, different parts of a problem can be done independently, so if you are unsure how to handle one part of a problem, don’t hesitate to try the others. No answer should require a great deal of computation or a complicated proof. Please show your work, and briefly explain your reasoning: correct answers with no work or explanation will not receive full credit.

1. Let $X_1, \ldots, X_n$ be i.i.d. random variables with common density

$$f(x|\theta, \gamma) = \frac{\theta \gamma^\theta}{x^{\theta+1}} I(x \geq \gamma)$$

having parameters $\theta > 0$ and $\gamma > 0$. Note that $f(x|\theta, \gamma)$ is supported on $[\gamma, \infty)$.

a. Find the MLE of $\gamma$ when $\theta$ is fixed. Does the estimate depend on $\theta$?

b. Find the MLE of $\theta$ when $\gamma$ is set to the value you found above.

c. Describe the general form of the likelihood ratio test statistic for testing

$$H_0 : \theta = 1 \text{ and } \gamma \text{ is arbitrary} \quad \text{vs} \quad H_1 : \theta \neq 1 \text{ and } \gamma \text{ is arbitrary},$$

and write out the statistic using the results of a. and b. above. You need not simplify.

2. Let $X \sim \mathcal{N}_d(\mu, \Sigma)$ be a multinormal random vector. Define $Y = CX$ and $Z = DX$ where $C \in \mathbb{R}^{l \times d}$ and $D \in \mathbb{R}^{k \times d}$ are matrices.

a. What is the distribution of $Y$?

b. Find necessary and sufficient conditions on $C$ and $D$ under which $Y$ and $Z$ are independent. Justify your answer.
3. Let \( \mathcal{P} = \{ f(x|\theta) : \theta \in \Theta \} \) be a family of densities on a set \( \mathcal{X} \), and suppose that we are interested in estimating \( \theta \) from an observation \( X \in \mathcal{X} \) with \( X \sim f(x|\theta) \in \mathcal{P} \).

a. Define what is meant by an estimator and a loss function in this setting.

b. Define the risk function of an estimator.

c. Let \( \pi \) be a prior distribution on \( \Theta \). Define the Bayes risk of an estimator under \( \pi \).

d. Let \( \mathcal{D} \) be a family of estimators. Define what it means for an estimator to be admissible.

4. Let \( V \subseteq \mathbb{R}^d \) be a finite set of vectors \( v = (v_1, \ldots, v_d)^t \) with \( L = \max_{v \in V} ||v||_2 \), and let \( \varepsilon_1, \ldots, \varepsilon_d \) be independent sign variables with \( \mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2 \). In answering the questions below, you may appeal to results from the lectures, but explain your reasoning.

a. Bound the moment generating functions of \( \sum_{i=1}^{d} \varepsilon_i v_i \) in terms of the constant \( L \).

b. Use the MGF bound to get an upper bound on \( \mathbb{E} \left[ \max_{v \in V} \sum_{i=1}^{d} \varepsilon_i v_i \right] \).

c. Use the MGF bound to get an upper bound on \( \mathbb{P}(\max_{v \in V} \sum_{i=1}^{d} \varepsilon_i v_i > t) \) when \( t > 0 \).

5. Let \( X_1, X_2, \ldots \) be i.i.d. positive random variables with finite expectation. For \( n \geq 1 \) define \( S_n = X_1 + \cdots + X_n \).

a. Calculate \( \mathbb{E}(S_n/S_m) \) when \( m \geq n \).

b. Find a lower bound for \( \mathbb{E}(S_n/S_m) \) when \( n \geq m \).

6. Let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) be a convex function that is Lipschitz with constant \( L \), and let \( U \in \mathbb{R}^d \) be uniformly distributed in the unit ball \( B(1) = \{ x : ||x|| = 1 \} \). Fix \( \delta > 0 \) and define a new function \( g(x) := \mathbb{E}f(x + \delta U) \). You may assume that \( g \) is well-defined for each \( x \).

a. What is the expected value of \( U \)?

b. Is \( g \leq f \), \( g \geq f \), or neither? Justify your answer.

c. Is \( g \) convex, concave or neither?

d. Is \( g \) Lipschitz?

e. What can you say about \( \sup_x |g(x) - f(x)| \)?